Nonlinear & Adaptive Systems: A Building Block Approach; Application in Control of Uncertain Mechanical System Subjected to unknown Sinusoidal Disturbance

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Adaptive systems and their applications, especially in intelligent control, have attracted a lot of attention. Many paradigms and approaches have been proposed, most of them rely on analytical techniques that could be understood only by those mathematically gifted. Lyapunov based (including the now popular integrator back-stepping) and estimation based schemes used commonly in adaptive control are very good examples under this category. In this paper, we advance a new paradigm that we call the building block approach for Nonlinear and Adaptive Systems Design, and its application to a specific example from adaptive control. This approach minimizes the need for complex analytical techniques (or moves the complexity away from a macro or global level to a micro or local level) in analyzing and or designing such systems, and more importantly, gives us the ability to concentrate more on what we can achieve from such complex systems. We will first discuss the approach from a general perspective for Nonlinear and Adaptive systems and then discuss in detail its application to a specific problem from adaptive control. We will also provide different family of controllers and simulation results to show the easy with which different adaptation schemes can be arrived using this method.

Keywords: Nonlinear and adaptive systems; NLTV circuits and dynamics; Adaptive control.

1. Introduction

In this paper, we discuss nonlinear and adaptive systems and application in adaptive control from a new and interesting perspective. Instead of looking at such systems from the classical analytical approach (dynamics to mathematical tools for characterization / stability testing to simulation and implementation), we look at such systems (and in fact, all nonlinear time-varying, NLTV, or non-autonomous systems) from an implementation (as electrical circuits) perspective. Thus, the thrust of our work is based on answers to number fundamental questions: 1) What kind of electrical elements or building blocks are needed to build complex NLTV circuits and what are their properties? 2) What kind of elements we need to build adaptive systems in particular? 3) If the process of adaptation or learning is viewed from a circuits' perspective, what is the circuit realization of this problem? 4) More importantly, can we use such elements' definitions, circuits formed (on a piece of paper) from such elements, and the dynamics resulting from such circuits as ***templates*** for further design, something similar to reverse engineering? The answers to these questions provide an elegant framework to handle the design of many systems in the nonlinear and time-varying domain. In this paper, we apply it to adaptive control. The new paradigm is shown in Fig. 1. As can be seen from the figure, it is a bottom-up approach for system design, where we start with the elements (in fact, their I/O mappings) and proceed to form circuits and the dynamics in terms of the I/O mappings of the elements. Such generic equations with the appropriate conditions are then used in real-world applications. For example, we can use the method proposed here to design chaotic and synchronizing chaotic systems useful in secure communication.

This new approach makes the understanding of complex nonlinear phenomena and the design of complex NLTV systems much easier, and using this simple approach we can teach undergraduate students NLTV systems very easily, as this author has been doing for a number of years. We recognize that certain macro level properties of electrical circuits such as passivity and positive real (PR) functions have long been used in areas such as I/O stability, hyper-stability and positivity applied to model reference adaptive system design, and dissipative system concepts based on energy and Lagrangian and Hamiltonian equations. Here we work at a micro or element level that provides us the flexibility to tailor the circuit / the dynamics to the application of interest. Further, our method is similar to reverse engineering where we try to make a new system by studying the product from a competitor, except that we are not breaking any laws here. We learn from electrical circuits (formed on a piece of paper) and use that knowledge to design electrical as well as non-electrical systems for various applications. One such application is discussed in the following sections.

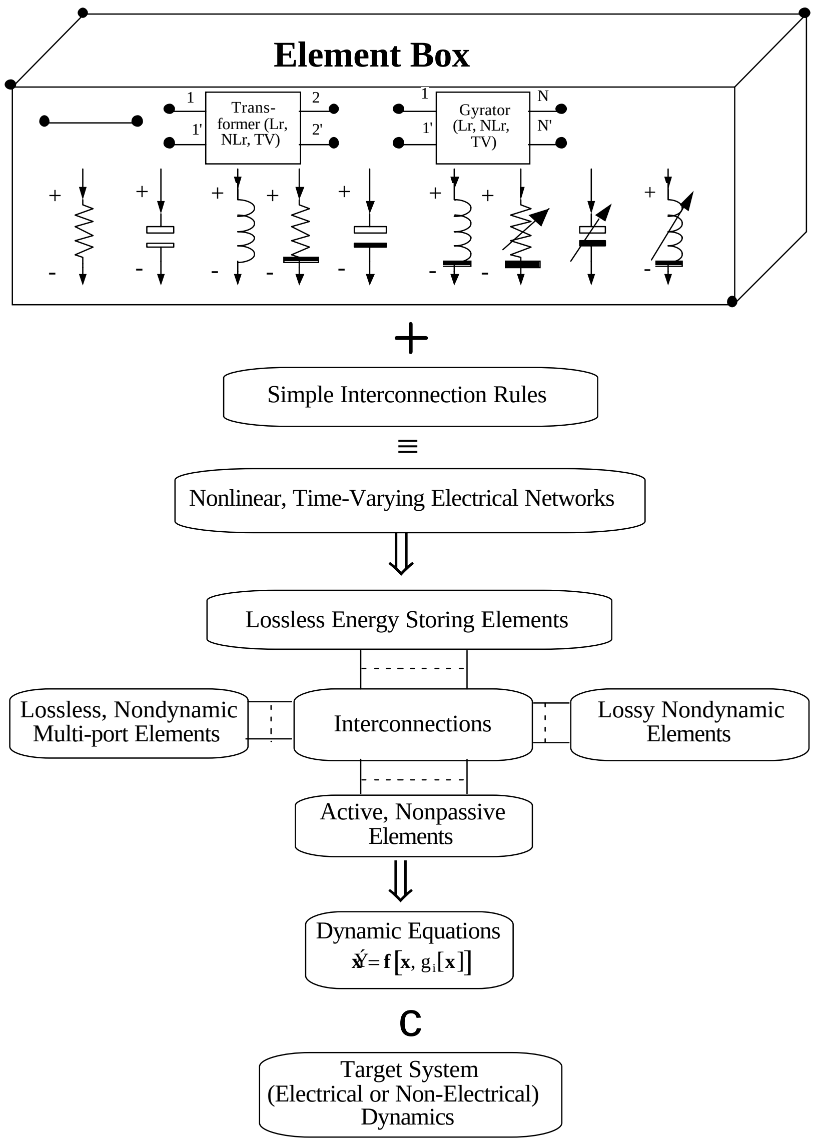

Figure 1. A new paradigm for nonlinear (autonomous & non-autonomous) dynamic systems design.

2. Nonlinear and Time-varying Circuit Elements:

In this section, we will briefly discuss the models for some NLTV electrical elements for building complex NLTV circuits[1]. The models will be arrived looking at physical realizability. Let us first consider the case of a *NLTV resistor*, a simple *static* device. We can write two generic models, one as *current controlled*, and another as *voltage controlled*.

$$v[t,\mathrm{a}] = v\left[i(t),\mathrm{a}\right] \qquad \text{or} \qquad i[t,\mathrm{b}] = i\left[v(t),\mathrm{b}\right] \tag{1}$$

Here the parameters a and b are used to lump the effects of the various physical phenomena that are assumed to change as a function of time. We can obtain different subcategories of NLTV resistors by looking at some possible realizations. For example, a *separable passive LTV model* can be obtained from a rheostat as:

$$v[t,\mathrm{a}] = R\left[\mathrm{a}(t)\right]i(t) \quad \text{with} \quad 0 < R_{\min} < R[t] < R_{\max} \tag{2}$$

where $R[\mathrm{a}]$ represents the time varying resistance and points to a resistor that is passive and consumes some minimum amount of power at any given time. We can make it nonlinear by letting the current to voltage relationship nonlinear. For example, we can have *separable* and *non-separable passive NLTV resistor* models:

$$v\left[i(t),\mathrm{a}\right] = R[\mathrm{a}]i^3(t) \tag{3}$$

$$v\left[i(t),\mathrm{a}\right] = i(t)\left\{i^2(t)+2i(t)\sin(t)+1\right\} \tag{4}$$

From the examples, we can note that $v\left[i(t)=0,\mathrm{a}\right]=0$ for all values of a and $v\left[i(t),\mathrm{a}\right]>0$ for $i(t)>0$ and $v\left[i(t),\mathrm{a}\right]<0$ for $i(t)<0$ for *passive resistors*. The characteristics will be in the second and fourth quadrants of the $v(t)$ Vs $i(t)$ plane for *negative resistors* and can be in any quadrant for *non-passive resistors*. Also, note that the mappings can be many to one and hence we need to maintain the input and output designations. The type of resistors, current-controlled or voltage-controlled, that we use in a circuit will depend on the reactive elements and the type of connection. We will discuss this issue when we deal with reactive elements.

A NLTV gyrator is a static M-port (M 2) element with the I/O relationship (given below for M = 2):

[1] Concepts such as power and energy are unique to the analog world. Hence, the analog or continuous-domain modeling is used here basically to arrive at the proper I/O mappings of various devices. Implementation of such devices and circuits is not an easy task as can be seen from the experience of researchers who worked on analog computers. Luckily, digital implementation has become the preferred mechanism for most applications, and that is what we propose here. We will derive the required dynamics in the analog domain, and transform it to the digital domain using proper A/D transformations. We can also preserve the basic properties such as lossless more easily in the digital domain even when finite precision arithmetic is used.

$$\begin{bmatrix} v_1[t,\alpha] \\ v_2[t,\alpha] \end{bmatrix} = \begin{bmatrix} 0 & R_{12}[\mathbf{x},\beta] \\ -R_{12}[\mathbf{x},\beta] & 0 \end{bmatrix} \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} = \mathbf{R}[\mathbf{x},\beta] \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} \tag{5}$$

Here, $\mathbf{R}[\mathbf{x},\beta]$ is the resistive matrix of the gyrator and is a function of time as well as the state vector **x**. Note that $\mathbf{R}[\mathbf{x},\beta] + \mathbf{R}^t[\mathbf{x},\beta] = \mathbf{0}$, which makes the power consumed by the gyrator zero all the time. That is, a NLTV gyrator is a *lossless static multi-port device* with a skew-symmetric resistor matrix $\mathbf{R}[\mathbf{x},\beta]$. Once again by proper choice of the matrix elements, we can arrive at specific classes of LTV, NLTI, separable, and non-separable models. The elements of the resistor matrix could be selected based on the application and normally be bounded for finite values of the state.

NLTV Inductors: We can visualize a NLTV inductor by considering its realization and letting some of the parameters associated with the realization vary as a function of time subject to some physical constraints. For modeling purposes, we can lump the effects of the change in various parameters into a single parameter α. We can have a current-controlled inductor, $\phi(t) = \phi[i_L(t),\alpha]$, or a flux-controlled inductor, $i_L(t) = i_L[\phi(t),\alpha]$. In either case, we recognize that at any given time instance, the NLTV inductor represents a NLTI inductor, and hence has to have all the properties of a inductor such as losslessness, energy storage and return (memory property). Hence considering a current controlled NLTV inductor, with for example $\phi_{Lr} = 0$ as the *relaxation point* (the value at which there is no energy left in the inductor), we get:

$$E_L[\phi_L = 0,\alpha] = 0 \quad \text{and} \quad E_L[\phi_L \neq 0,\alpha] > 0 \quad \text{for all } \alpha \tag{6a}$$

$$E_{LL}[\phi_L] \le E_L[\phi_L,\alpha] \le E_{Lu}[\phi_L] \tag{6b}$$

where $E_{LL}[\phi_L]$,and $E_{Lu}[\phi_L]$ are two positive definite functions of the variable ϕ_L. We arrive at the first constraint because no flux implies no stored energy regardless of what values the physical parameters take. The second constraint follows from the definition of stored energy. The third condition says the energy function of a NLTV inductor must be a locally positive definite and a decrescent function[2]. That is, the stored energy has to be bounded by some positive definite functions of the flux alone and follows from the fact that as the values of the parameters are bounded, and for any fixed value of α or time t, the device represents an inductor with bounded energy for bounded ϕ_L. From the energy expression, we can obtain an expression for the power (entering the element) as $p_{L\_in} = dE_L/dt$ and the current as $i_L(t) = \partial E_L[\phi_L,\alpha]/\partial\phi_L$. Similar to the case of resistors, we can obtain LTV, NLTI, separable, and non-separable models. Some examples are:

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[2] We can let $E_L[\phi_L,\alpha] \to \infty$ as $|\phi_L| \to \infty$ as in the case of LTI inductors so that we can talk of the properties in a global sense.

Separable Linear time-varying inductor:

$$
\begin{aligned}
E_L\left[\varphi_L(t),\alpha(t)\right] &= \left(1+\sin^2(t)\right)\varphi_L^2(t)\,; \\
p_{L\_in} &= \underbrace{2\left(1+\sin^2(t)\right)\varphi_L(t)}_{i_L(t)} \;*\; \underbrace{\left(\dot{\varphi}_L(t)+0.5\varphi_L(t)\sin(2t)\right)}_{v_{L\_flux}+v_{L\_TV\_Mechanism}}
\end{aligned}
\tag{7}
$$

where $\dot{\varphi}_L(t)=v_{L\_flux}$ is the voltage resulting due to changing flux and $v_{L\_TV\_Mechanism}=0.5\varphi_L(t)\sin(t)$ is the voltage that results dues to the time-varying mechanism when the flux is constant.

Separable Nonlinear TV inductor:

$$
\begin{aligned}
E_L\left[\varphi_L(t),\alpha(t)\right] &= \left(1+\sin^2(t)\right)\varphi_L^4(t) \\
p_{L\_in} &= 4\left(1+\sin^2(t)\right)\varphi_L^3(t)\left(\dot{\varphi}_L(t)+0.5\varphi_L(t)\sin(2t)\right) \\
i_L(t) &= 4\left(1+\sin^2(t)\right)\varphi_L^3(t)
\end{aligned}
\tag{8}
$$

Non-separable NLTV inductor with relaxation points at $\varphi_{Lr}=0$.:

$$
\begin{aligned}
E_L\left[\varphi_L(t),\alpha(t)\right] &= \varphi_L^2\left(\varphi_L^2-2\varphi_L\sin(2t)+2\right) = \varphi_L^2\left(\left(\varphi_L-\sin(2t)\right)^2+1+\cos^2(2t)\right) \\
&= \varphi_L^2\, f(\varphi_L,t)\,; \quad f(\varphi_L,t)>0 \;\; \textit{for all}\;\; \varphi_L,t \\
p_{L\_in} &= 2\varphi_L\, f(\varphi_L,t)\left(\dot{\varphi}_L-\frac{2\varphi_L^2\cos(2t)}{f(\varphi_L,t)}\right)
\end{aligned}
\tag{9}
$$

We can have *multiple relaxation points* if consider *NLTI inductors*. For example, we can have:

$$
\begin{aligned}
E_L\left[\varphi_L(t)\right] &= 0.25\left(\varphi_L^2-1\right)^2 \\
p_{L\_in} &= \varphi_L\left(\varphi_L^2-1\right)\dot{\varphi}_L = i_L\left[\varphi_L(t)\right]*v_L(t)
\end{aligned}
\tag{10}
$$

which implies we have a many to one flux to current mapping $i_L\left[\varphi_L(t)\right]$. It appears we cannot define such an element in the NLTV case and hence the NLTV elements should be restricted to one to one (invertible) mapping. Finally, it should be pointed out that when a NLTV resistor is connected in series with an inductor, we need to make the resistor current controlled to ensure the solvability of the resulting dynamics.

A *time-varying capacitor*, being the dual of an inductor, can be defined in a similar manner. We can also define *ideal NLTV transformers* (similar to ideal LTI transformers) and *NLTV mutual or multi-port inductors and capacitors*. We will omit the details here. Ref. [5] contains additional information.

3. NLTV Electrical Circuits, Dynamics, and Resulting Responses

We can form various circuits from the building blocks defined in the previous section and obtain the general form of NLTV dynamics from the circuits. It should be clear that the dynamics will have nonlinear and time-varying terms that represent the mappings of valid electrical elements. By varying the mappings within the allowed domain for each of the elements, we can obtain all possible NLTV dynamics belonging to a particular family. Note that the reactive elements still play the role of energy storage and return (lossless, memory devices), but the energy envelope will be more complex than just the bowel shaped energy curve of LTI memory elements. Further, they can have multiple relaxation or zero energy points and local minima and maxima points. The relaxation points and the points of local minima of the energy become the equilibrium points of the dynamics. The sum of the energy in all the reactive elements in the circuit is one good Lyapunov function. The total energy function resulting from the use of only single-port inductors and or capacitors will be a separable function of the state variables. The use of mutual or multi-port inductors will lead to non-separable energy functions. The lossless, static multi-port Elements such as gyrators and transformers play the role of transferring power from some ports to the rest of the ports, and hence help to shape the required response. The NLTV resistors provide the necessary mechanism to consume power (if that is what is needed as in a stable dynamics) or could be used to consume and or deliver power depending on the value of the independent variable (as in limit cycle oscillations or chaotic response). We will now look at some general circuits made of specific category of elements and state the properties of resulting dynamics / their equilibrium points[3],

Type 1 Circuits: NLTV circuits made of NLTV reactive elements, static multi-port lossless elements, and resistors that are globally passive (that is, the voltage to current characteristic is confined to the first and the third quadrant). Further, let the reactive elements have their relaxation points at the origin only and have monotonically increasing energy storage functions with the energy becoming unbounded as the independent variables become unbounded: $\Rightarrow$ The resulting dynamics will have only one equilibrium point at the origin which will be *globally uniformly asymptotically stable* (UAS). A Lyapunov function for the dynamics will be the sum of energy stored in the various reactive elements. The derivative of the LF along the system trajectory will be the negative of the sum of power consumed by the resistors, and will always be negative for globally passive resistors. The shape of the resistor characteristics determines whether the equilibrium point is \textit {exponentially} stable or not, and if the dynamics is *totally stable* (stability under persistent disturbances). Couple of simple examples will illustrate these points. Consider a circuit with a single LTI capacitor of value 1 F, a NLTV voltage controlled resistor given by $i_R\left[v_R(t), t\right]=\left(1+\sin^2(t)\right)v_R^3(t)$ and a current source $i_s(t)$ all connected in parallel. Using KCL & KVL, and the elements' equations, we get circuit dynamics as: $\dot{v}_c(t)=-\left(1+\sin^2(t)\right)v_R^3(t)+i_s(t)$. We can note that this dynamics (with $i_s(t)=0$) has only one equilibrium point at the origin that is globally UAS. However, the equilibrium point is not exponentially stable as the power consumption capacity of the resistor becomes negligible $\left(p_R(t)=\left(1+\sin^2(t)\right)v_R^4(t)=\left(1+\sin^2(t)\right)v_c^4(t)\right)$ when $\left|v_c(t)\right|<1$. Similarly, if we replace the resistor by one with a characteristic $i_R\left[v_R(t), t\right]=\tanh\left[v_R(t)\right]$ the resulting dynamics will have the origin as the only exponentially stable equilibrium point. However, we can note that the dynamics

[3] Given the complexity of NLTV elements and the circuits that can be made, it should be noted that the properties stated here are valid from a practical or conservative design perspective. However, we can also arrive at dynamics which have the same properties, but pointing to a different class of circuits. As an example, we can combine a number of non-passive resistors to come up with the characteristics of a globally passive resistor circuit.

will not be totally stable since the power absorption capacity is reduced to $p_R(t) = |v_R(t)| = |v_c(t)|$ as $|v_c(t)| >> 1$. These two simple examples demonstrate the power of looking at NLTV dynamics not just from a mathematical perspective but as one from a circuit made of real elements.

Type 2 Circuits: NLTV circuits made of NLTV reactive elements with *multiple relaxation points and or multiple local minimal energy points*, static multi-port lossless elements, and resistors that are globally passive: $\Rightarrow$ The resulting dynamics will have multiple equilibrium points, some of which will be locally UAS, and some will be unstable. Whether the stable equilibrium points are exponentially and or totally stable will depend on the resistor characteristics.

Type 3 Circuits: NLTV circuits made of NLTV reactive elements with *multiple relaxation points and or multiple local minimal energy points*, static multi-port lossless elements, and resistors whose properties change from passive to non-passive or vice versa based on the input values $\Rightarrow$ The properties of the resulting dynamics can be all over the map depending on when the resistors become passive and non-passive. These types of circuits can be configured easily to arrive at dynamics that exhibit limit cycle and chaotic behavior.

4. Application To Adaptive Control

In order to explain the underlying concept, we use a single-degree-of-freedom (SDOF) linear mass-spring-damper trio, subjected to a simple sinusoidal disturbance $F_d(t)$ (see Fig. 1(*a*)) considered in ref. [1, 2]. Assuming linear suspension and neglecting nonlinear friction effects, the system is represented by

$$\begin{aligned} M\ddot{y}(t) + C\dot{y}(t) + k\,y(t) &= u(t) + F_d(t) \\ F_d(t) &= Q\sin(\omega t) \end{aligned} \qquad (1)$$

where $y(t) \in \mathbb{R}$ is the mass position, $M, C, k \in \mathbb{R}$ represent the mass quantity, damping coefficient and spring constant, respectively, $u(t) \in \mathbb{R}$ denotes the input force, and $F_d(t) \in \mathbb{R}$ is the disturbance signal with Q, ω representing its constant amplitude and frequency, respectively. For the system given by Eq. (1), $y(t)$ is the only measurable signal and all the system parameters, including the amplitude and frequency of the disturbance $F_d(t)$ are considered to be unknown constants.

This second order model containing the nonlinear term $\sin(\omega t)$ can be made into a fourth order linear model (in terms of the parameters) by differentiating it twice and adding that to the original equation scaled by the factor ω^2. The resulting fourth order model can be written in state space form as:

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\,\mathbf{x} + \mathbf{F}[y,u]\theta \\
y &= \mathbf{e}_1^t\,\mathbf{x}
\end{aligned}
\tag{2}
$$

where $\mathbf{x}(t) \in \mathbb{R}^4$ is the state vector, and matrix $\mathbf{A}$ is defined as

$$
\mathbf{A} = \begin{bmatrix} 0 & \vdots & \mathbf{I}_3 \\ 0 & \vdots & 0 \end{bmatrix} \equiv \begin{bmatrix} \mathbf{A}_1 & \vdots & \mathbf{A}_2 \end{bmatrix} \tag{3}
$$

with $\mathbf{I}_3 \in \mathbb{R}^{3\times3}$ representing the corresponding identity matrix and the matrix $\mathbf{A}$ is expressed in terms of sub-matrices associated with the known state variable $x_1(t)$ and the rest. $\mathbf{F}[y,u] \in \mathbb{R}^{4\times7}$ is defined as

$$
\mathbf{F}[y,u] = \begin{bmatrix} \begin{bmatrix} 0_{1\times3} \\ \mathbf{I}_3 \end{bmatrix} u & -\mathbf{I}_4 y \end{bmatrix} \tag{4}
$$

with $\mathbf{I}_4 \in \mathbb{R}^{4\times4}$ representing the corresponding identity matrix, $\mathbf{e}_1 = [1 \quad 0 \quad 0 \quad 0]^t$ and $\theta \in \mathbb{R}^7$ denotes the constant unknown parameters vector:

$$
\begin{aligned}
\theta &= \begin{bmatrix} b_2 & b_1 & b_0 & a_3 & a_2 & a_1 & a_0 \end{bmatrix}^t \\
&\equiv \begin{bmatrix} \frac{1}{M} & 0 & \frac{\omega^2}{M} & \frac{C}{M} & \omega^2 + \frac{k}{M} & \frac{C\omega^2}{M} & \frac{k\omega^2}{M} \end{bmatrix}^t
\end{aligned}
\tag{5}
$$

This particular state space representation is used to separate the effect of the various parameters on the dynamics from that of the state as will become obvious later (put in the equation number). We have also introduced a parameter b1 with a value equal to zero to introduce the input $u(t)$ in to the dynamics of $x_3(t)$.

If we assume that the various parameters are known and only the state variable $x_1(t) \equiv y(t)$ is given and we need to estimate the other three state variables $x_2(t)$ *to* $x_4(t)$, we can write a general form of the estimator dynamics as:

$$
\dot{\hat{\mathbf{x}}} = \mathbf{f}_{est}[\hat{\mathbf{x}}, x_1, u] \tag{6}
$$

where $\hat{\mathbf{x}}$ is the estimated state vector. Note that the estimator dynamics use the only known state variable $x_1(t)$ and the known input $u(t)$. Since this is a LTI system, we can design the estimator easily using known analytical techniques. Let us express the dynamics in terms of the error vector $\varepsilon = \mathbf{x} - \hat{\mathbf{x}}$ as:

$$
\dot{\varepsilon} = \mathbf{f}_{error}[\varepsilon] = \mathbf{A}_e\,\varepsilon \tag{7a}
$$

and $\mathbf{A}_e$ is a 4x4 matrix in our case. The error dynamics should have the origin as the globally stable equilibrium point implying that for any non-zero initial error vector $\boldsymbol{\varepsilon}(0) \neq [0]$, $\boldsymbol{\varepsilon}(t) \rightarrow [0]$ or $\hat{\mathbf{x}}(t) \rightarrow \mathbf{x}(t)$ as $t \rightarrow \infty$. Equivalently, the polynomial given by $\det(s\mathbf{I} - \mathbf{A}_e)$ must be strictly Hurwitz. From a circuits' perspective we can state that the error dynamics should correspond to a passive circuit with reactive elements whose relaxation points (where they have no stored energy) are at the origin. Further the energy storage capability of the reactive elements should be a monotonically increasing function of the independent variables. For LTI circuits, the general form of such a circuit would be a n-port coupled-inductor (a lossless reactive element with memory) described by the positive definite inductance matrix $\mathbf{L}$ connected to a n-port static passive network with the impedance matrix $\mathbf{Z}_p$. Further, $\mathbf{Z}_p = \mathbf{Z}_R + \mathbf{Z}_a$, with $\mathbf{Z}_R$ a semi positive definite (PD) matrix corresponding to a lossy resistor network, and $\mathbf{Z}_a$ an anti-metric matrix corresponding to a lossless network known as gyrator. Using KVL, KCL, the elements' equations and the architecture described, we can write the dynamics of the passive circuit as[4]:

:

$$\dot{\boldsymbol{\varepsilon}} = -\mathbf{L}^{-1}\mathbf{Z}_p\boldsymbol{\varepsilon} \tag{7b}$$

We also noted above that the resulting estimator dynamics should have a particular form. That is, it should involve only the known state variable x_1. This imposes some constraints on the error dynamics. From the error dynamics and the plant dynamics, we can write the estimator dynamics as:

$$\begin{aligned}
\dot{\hat{\mathbf{x}}} &= \dot{\mathbf{x}} - \dot{\boldsymbol{\varepsilon}} = \mathbf{A}\mathbf{x} + \mathbf{F}[y,u]\theta - \mathbf{A}_e\boldsymbol{\varepsilon} \\
&= [\mathbf{A}_1 \vdots \ \mathbf{A}_2][x_1 \vdots \ x_2 \quad x_3 \quad x_4]^t + \mathbf{F}[y,u]\theta - [\mathbf{A}_{e1} \vdots \ \mathbf{A}_{e2}][\varepsilon_1 \vdots \ \varepsilon_2 \quad \varepsilon_3 \quad \varepsilon_4]^t \\
&= \mathbf{A}_1 x_1 - \mathbf{A}_{e1}(x_1 - \hat{x}_1) + \mathbf{F}[y,u]\theta + \mathbf{A}_{e2}[\hat{x}_2 \quad \hat{x}_3 \quad \hat{x}_4]^t + [\mathbf{A}_2 - \mathbf{A}_{e2}][x_2 \quad x_3 \quad x_4]^t \\
&= \mathbf{A}_e\hat{\mathbf{x}} + \mathbf{F}[y,u]\theta + \mathbf{k}x_1 \qquad \textit{if} \ \ \mathbf{k} = \mathbf{A}_1 - \mathbf{A}_{e1} \ \textit{and} \ \mathbf{A}_{e2} = \mathbf{A}_2
\end{aligned} \tag{8}$$

That is, the sub-matrix $\mathbf{A}_{e2}$ should be constrained to be same as the plant sub-matrix $\mathbf{A}_2$ and hence we should know $\mathbf{A}_2$ exactly. That should not be surprising since the sub-state-vector $[x_2 \quad x_3 \quad x_4]^t$ is

---

[4] Comparing 7a and 7b, we can note that $\mathbf{A}_e = -\mathbf{L}^{-1}\mathbf{Z}_p$. To calculate the matrix pair $\{\mathbf{L}, \mathbf{Z}_p\}$ given $\mathbf{A}_e$, we will assume a semi PD matrix $\mathbf{Z}_R$ and use Kalman-Yakubovic Theorem to obtain $\mathbf{L}$ from $-0.5\left(\mathbf{L}\mathbf{A}_e + \mathbf{A}_e^t\mathbf{L}\right) = \mathbf{Z}_R$, which should come out to be PD for any stable system matrix $\mathbf{A}_e$. Then we find $\mathbf{Z}_p$ from $\mathbf{Z}_p \equiv \mathbf{Z}_a + \mathbf{Z}_R = -\mathbf{L}\mathbf{A}_e$. That is, once we fix $\mathbf{Z}_R$, the value $\mathbf{L}$ and hence $\mathbf{Z}_p$ gets automatically fixed for any given system.

unknown, we should at least know the architecture corresponding to those state variables. So, only $\mathbf{A}_{e1}$ is at our disposal to make the error dynamics stable. Since we are dealing with a LTI error dynamics, the choice $\mathbf{A}_{e1} \equiv -\mathbf{k} = \begin{bmatrix} -k_1 & -k_2 & -k_3 & -k_4 \end{bmatrix}^t$ with certain range of values for k_i can indeed make the error dynamics stable, or equivalently we can obtain an estimator where $\hat{\mathbf{x}}(t) \to \mathbf{x}(t)$ as $t \to \infty$.

We are interested here in the more complex problem of tracking control ($y(t) \to y_d(t)$ as $t \to \infty$, where $y_d(t)$ is the desired output) while learning the parameters in an adaptive manner and keeping all the signals bounded. If we measure all the state variables of the state vector $\mathbf{x}$ and $\mathbf{x}_d$ is the given or the desired trajectory, we could come up with a simpler adaptive algorithm where we estimate the parameters while forcing the plant output to track the desired output. That is not the case here. Hence we will use additional error terms for tracking in addition to the state estimation errors ε_i $(i = 1\ to\ 4)$. Let us first define a tracking error $z_1(t) = y(t) - y_d(t)$, and another error $z_2(t)$, to be defined later as a function of $\dot{y}_d(t)$, and $z_2(t) \to 0$ as $\dot{y}(t) \to \dot{y}_d(t)$. Let us also modify the estimator and the error dynamics to include these new error variables and write them using circuits' interpretation[5] as:

$$\mathbf{L}\dot{\boldsymbol{\varepsilon}} = -\mathbf{Z}_p\,\boldsymbol{\varepsilon} - \mathbf{f}\left[z_1, z_2\right] \tag{9a}$$

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}_e\,\hat{\mathbf{x}} + \mathbf{F}\left[y,u\right]\boldsymbol{\theta} + \mathbf{k}\,y + \mathbf{L}^{-1}\mathbf{f}\left[z_1, z_2\right] \tag{9b}$$

where $\mathbf{f}\left[z_1(t), z_2(t)\right]$ $(= \left[\, fn_{i1}\, z_1 + fn_{i2}\, z_2 \right]$, i = 1 to 4 and fn_{ij} could be constants or functions of the various error variables) is a suitable vector function to be defined later. This estimator still cannot be implemented as it involves the unknown parameter-vector $\boldsymbol{\theta}$. Instead, we will implement a part of this system as two separate filters based respectively on the measured signal y(t) and the input $u(t)$. Let us first define $\hat{\mathbf{x}}$ as a function of a new vector $\boldsymbol{\xi}(t) \in \mathbb{R}^4$ and a matrix $\Omega(t) \in \mathbb{R}^{4\times 7}$ as:

$$\hat{\mathbf{x}} = \boldsymbol{\xi} + \Omega\boldsymbol{\theta} \tag{10}$$

Then from the estimator dynamics in (9b) we obtain the following dynamic equations to calculate $\boldsymbol{\xi}(t)$ and $\Omega(t)$:

$$\dot{\boldsymbol{\xi}} = \mathbf{A}_e\,\boldsymbol{\xi} + \mathbf{k}\,y + \mathbf{L}^{-1}\mathbf{f}\left[z_1, z_2\right] \tag{11a}$$

$$\dot{\Omega} = \mathbf{A}_e\,\Omega + \mathbf{F}\left[y,u\right] \tag{11b}$$

[5] Unlike the analytical approach based on backtracking, we have included the error vector $\mathbf{f}\left[z_1, z_2\right]$ also to arrive a fully coupled dynamics as will become clear later.

From equations (10) and (11), we can write the state estimation error and the state in terms of these known vectors as:

$$\varepsilon = \mathbf{x} - \hat{\mathbf{x}} \tag{12a}$$

$$\mathbf{x} = \xi + \Omega\theta + \varepsilon \tag{12b}$$

We will use the above equation for the state to arrive at the tracking / adaptation equations. Our task is to define an augmented error vector **E** involving the state error vector ε the tracking error variables z_1, z_2, the parameter error vector $\tilde{\theta} = \theta - \hat{\theta}$ (where $\hat{\theta}$ is the estimated parameter vector θ) and an additional parameter error $\tilde{\rho} = \rho - \hat{\rho}$ (where $\rho = 1/b_2$ and $\hat{\rho}$ is the estimate of ρ) and arrive at the augmented error dynamics using the only available input $u(t)$ and simultaneous add and subtract (of the same terms) operations. That is:

$$\mathbf{E} = \left[z_1 \quad z_2 \quad \varepsilon_1 \quad \varepsilon_2 \quad \varepsilon_3 \quad \varepsilon_4 \quad \tilde{\rho} \quad \tilde{\theta}_{1\times 7} = \left(\tilde{\theta}_1 \cdots \tilde{\theta}_7\right)\right]^t \in \mathbb{R}^{14} \tag{13a}$$

$$\dot{\mathbf{E}} = \mathbf{f}\left[\mathbf{E}\right] \tag{13b}$$

The dynamics should have the origin as the globally uniformly asymptotically stable (UAS) equilibrium point or should point to a passive (LTI to NLTV) circuit as discussed before. We will use the knowledge derived from various passive (LTI, LTV, NLTI and NLTV) circuit architectures to solve this problem with out resorting to complex analytical techniques such as selecting Lyapunov functions candidates, evaluating their derivatives and so on. Since we have only one input at our disposal we will also have to use add / subtract techniques. First let us consider expressing the derivative of the tracking error $z_1(t)$ in terms of the various error variables. We can start with the first two equations of the plant dynamics:

$$\dot{x}_1 = x_2 - a_3\, y \tag{14a}$$

$$\dot{x}_2 = x_3 - a_2\, y + b_2\, u \tag{14b}$$

and write $\dot{z}_1(t)$ as:

$$\dot{z}_1(t) = \dot{y} - \dot{y}_d = x_2 - a_3\, y - \dot{y}_d \tag{15}$$

which should be made a function of the error vector **E**. But that is not directly possible since we do not have x_2 and a_3, and there is no input term to manipulate it any further. Hence we will express the state variables in terms of the known components ξ, Ω belonging to the state estimate. We first rewrite the matrix Ω as:

$$\Omega = \begin{bmatrix} \mathbf{v}_2 & \mathbf{v}_1 & \mathbf{v}_0 & \mathbf{f}_3 & \mathbf{f}_2 & \mathbf{f}_1 & \mathbf{f}_0 \end{bmatrix} \tag{16}$$

where $\mathbf{v}_i, \mathbf{f}_i \in \mathbb{R}^4$ are column vectors. Let v_{ji} be the i-th element of the column vector $\mathbf{v}_j$ (similarly for $\mathbf{f}_j$). We can rewrite (14a) using (11) and (16) as:

$$\dot{x}_1 = \dot{y} = x_2 - a_3 y = \xi_2 + \omega\theta + \varepsilon_2 = \xi_2 + b_2 v_{22} + \bar{\omega}\theta + \varepsilon_2 \tag{17a}$$

where

$$\omega = \begin{bmatrix} v_{22} & v_{12} & v_{02} & f_{32} - y & f_{22} & f_{12} & f_{02} \end{bmatrix} \tag{17b}$$

$$\bar{\omega} = \begin{bmatrix} 0 & v_{12} & v_{02} & f_{32} - y & f_{22} & f_{12} & f_{02} \end{bmatrix} \tag{17c}$$

Here we kept the variable v_{22} separately in the right most expression in (17a) since it is available and from (11b) and (16), its derivative is seen to be connected to the input $u(t)$ as:

$$\dot{v}_{22} = -k_2 v_{21} + v_{23} + u \tag{18}$$

From equation (17), we can write the derivative of the tracking error as:

$$\dot{e}_1 = \dot{y} - \dot{y}_d = \xi_2 + b_2 v_{22} + \bar{\omega}\theta + \varepsilon_2 - \dot{y}_d \tag{19}$$

Still this equation is not in terms of the various augmented error variables. And we do not have an input variable to accomplish this in a straight forward manner. Hence we need to achieve this by introducing new variables. Note $\varepsilon_2 = 0$ and $\dot{y}_d = \xi_2 + b_2 v_{22} + \bar{\omega}\theta$ when tracking is achieved. Using the estimated parameters, this expression could be written as:

$$\frac{1}{\hat{b}_2}\dot{y}_d \equiv \hat{\rho}\,\dot{y}_d = v_{22} + \hat{\rho}\left(\xi_2 + \bar{\omega}\hat{\theta}\right) + e_2 \tag{20}$$

where as noted before, $\hat{r}$ the estimate of the parameter $r = 1/b_2$ and e_2 denotes the error due to the use of the estimated parameters. Now let us define

$$a_1 = \hat{r}\left(x_2 + \bar{w}\hat{q} + m_{11} z_1\right);\ \bar{a}_1 = a_1/\hat{r} \tag{21}$$

with m_{11} a known positive constant and where a_1 can be calculated. Equation (20) becomes:

$$\hat{r}\,\dot{y}_d = v_{22} + a_1 - \hat{r}\, m_{11} z_1 + e_2 \equiv v_{22} + a_1 - z_2 \tag{22}$$

where $z_2 \equiv \hat{r}\, m_{11} z_1 - e_2 = v_{22} + a_1 - \hat{r}\,\dot{y}_d$ is a new error variable that depends on $\dot{y}_d$ and goes to zero as tracking is achieved. Using a_1 in lieu of $\dot{y}_d$ equation (19) can be rewritten as:

$$\dot{z}_1 = -m_{11} z_1 + \hat{b}_2 z_2 + e_2 + w\tilde{q} \tag{23}$$

Equation (23) is one of the augmented error dynamics; it is still not complete in the sense it doesn't involve the parameter error variable $\tilde{r}$ (we will need this so as to arrive at the adaptive equation for $\hat{r}$ as we will see shortly). Luckily this can be rectified by noting that $b_2\tilde{r} + \hat{r}\tilde{b}_2 = 0$ and introducing a weighted value of it in the above equation (we use $-\left(b_2\tilde{r} + \hat{r}\tilde{b}_2\right)\left(\dot{y}_d - \bar{a}_1\right)$ as in the original paper though any weight could have been used) to arrive at:

$$\dot{z}_1 = -m_{11} z_1 + \hat{b}_2 z_2 + e_2 - b_2\left(\dot{y}_d - \bar{a}_1\right)\tilde{r} + \left(w - \hat{r}\left(\dot{y}_d - \bar{a}_1\right)\mathbf{e}_1^t\right)\tilde{q} \tag{24}$$

This forms our first error dynamic equation involving the augmented error vector: **E**. Let us put this dynamic equation into a complete error dynamics in the state space form as[6]:

[6] We show the complete dynamics here and use it in our further discussions on how subsequent dynamic equations are obtained. For later discussion purposes, let us denote the vector on the LHS as $\hat{\mathbf{L}}\dot{\mathbf{E}}$ and the matrix on the RHS as $-\hat{\mathbf{Z}}_p$.

$$
\begin{bmatrix} \dot z_1 \\ c_2 \dot z_2 \\ \Sigma_{j=1}^4 L_{1j}\dot\varepsilon_j \\ \Sigma_{j=1}^4 L_{2j}\dot\varepsilon_j \\ \Sigma_{j=1}^4 L_{3j}\dot\varepsilon_j \\ \Sigma_{j=1}^4 L_{4j}\dot\varepsilon_j \\ c_7 \dot{\tilde\rho} \\ \bar{\mathbf{L}}\,\dot{\tilde\theta}_{7\times1} \end{bmatrix}
=
\begin{bmatrix}
-m_{11} & m_{12} & 0 & 1 & 0 & 0 & m_{17} & (\bar m_1)_{1\times7} \\
-m_{12} & -m_{22} & 0 & m_{24} & 0 & 0 & 0 & (\bar m_2)_{1\times7} \\
-fn_{11} & -fn_{12} & -z_{p11} & -z_{p12} & -z_{p13} & -z_{p14} & 0 & 0_{1\times7} \\
-fn_{21} & -fn_{22} & -z_{p21} & -z_{p22} & -z_{p23} & -z_{p24} & 0 & 0_{1\times7} \\
-fn_{31} & -fn_{32} & -z_{p31} & -z_{p32} & -z_{p33} & -z_{p34} & 0 & 0_{1\times7} \\
-fn_{41} & -fn_{42} & -z_{p41} & -z_{p42} & -z_{p43} & -z_{p44} & 0 & 0_{1\times7} \\
-m_{17} & 0 & 0 & 0 & 0 & 0 & 0 & 0_{1\times7} \\
-(\bar m_1^t)_{1\times7} & -(\bar m_2^t)_{1\times7} & 0_{7\times1} & 0_{7\times1} & 0_{7\times1} & 0_{7\times1} & 0_{7\times1} & 0_{7\times7}
\end{bmatrix}
\begin{bmatrix} z_1 \\ z_2 \\ \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \tilde\rho \\ \tilde\theta \end{bmatrix}
$$

$$
\left[\leftarrow - \hat{\mathbf{L}}\dot{\mathbf{E}} - \rightarrow\right] = \left[\leftarrow\text{------------------}\quad -\hat{\mathbf{Z}}_p \quad\text{------------------}\rightarrow\right]\left[\leftarrow \mathbf{E} \rightarrow\right]
$$

25a)

$$
\begin{aligned}
& m_{11}, c_2, m_{22} > 0; \quad m_{12} = \hat b_2; \;\; m_{17} = -b_2\left(\dot y_d + \bar\alpha_1\right); \;\; \bar m_1 = \omega - \hat\rho\left(\dot y_d + \bar\alpha_1\right)\mathbf{e}_1^t \\
& m_{24} = c_2 \frac{\partial\alpha_1}{\partial y}; \;\; \bar m_2 = \left(c_2 \frac{\partial\alpha_1}{\partial y}\omega\right)_{1\times7} = m_{24}\,\omega; \\
& fn_{11} = fn_{12} = fn_{31} = fn_{32} = fn_{41} = fn_{42} = 0; \quad fn_{21} = 1; \quad fn_{22} = m_{24}; \\
& c_7 = \gamma\left|b_2\right| > 0; \;\; \bar{\mathbf{L}} \textit{ and } \hat{\mathbf{L}} \textit{ are PD matrices.}
\end{aligned} \tag{25b}
$$

Let us now consider the derivative of z_2 .From equations (18), (21) and (22), we can write it as:

$$
\dot z_2 = \dot v_{22} + \dot\alpha_1 - \hat\rho\,\ddot y_d - \dot{\hat\rho}\,\dot y_d = u(t) - k_2 v_{21} + v_{23} + \dot\alpha_1 - \hat\rho\,\ddot y_d - \dot{\hat\rho}\,\dot y_d \tag{26}
$$

Our aim is to bring this dynamics in terms of the error vector **E** through the use $u(t)$ with the constraint that $u(t)$ should involve only terms that are available and or can be calculated. Since $\alpha_1 = \alpha_1\left(y, y_d, \xi_2, \hat\theta, \hat\rho, v_{12}, v_{02}, f_{32}, f_{22}, f_{12}, f_{02}\right)$ we can write its derivative as:

$$
\dot\alpha_1 = \frac{\partial\alpha_1}{\partial y}\dot y + \frac{\partial\alpha_1}{\partial y_d}\dot y_d + \frac{\partial\alpha_1}{\partial\xi_2}\dot\xi_2 + \frac{\partial\alpha_1}{\partial\hat\theta}\dot{\hat\theta} + \frac{\partial\alpha_1}{\partial\hat\rho}\dot{\hat\rho} + \sum_{i=0,1}\frac{\partial\alpha_1}{\partial v_{i2}}\dot v_{i2} + \sum_{i=0}^{3}\frac{\partial\alpha_1}{\partial f_{i2}}\dot f_{i2} \tag{27}
$$

where using (15a) the first term on the RHS can be rewritten as:

$$
\frac{\partial\alpha_1}{\partial y}\dot y = \frac{\partial\alpha_1}{\partial y}\left(\xi_2 + \omega\theta + \varepsilon_2\right) = \frac{\partial\alpha_1}{\partial y}\left(\left(\omega\tilde\theta + \varepsilon_2\right) + \left(\omega\hat\theta + \xi_2\right)\right) \tag{28}
$$

From (26) to (28) we can separate the terms in the derivative of z_2 into a) the terms that are available as input or can be calculated, b) terms based on the error variables, and c) the terms that are unknown:

$$\dot z_2 = u(t) - \beta + \frac{\partial \alpha_1}{\partial y}\left(\omega\tilde\theta + \varepsilon_2\right) + \frac{\partial \alpha_1}{\partial \hat\theta}\dot{\hat\theta} + \frac{\partial \alpha_1}{\partial \hat\rho}\dot{\hat\rho} - \dot{\hat\rho}\,\dot y_d \tag{29a}$$

where

$$\beta = k_2 v_{21} - v_{23} + \hat\rho\,\ddot y_d - \frac{\partial \alpha_1}{\partial y}\left(\omega\hat\theta + \xi_2\right) - \frac{\partial \alpha_1}{\partial y_d}\dot y_d - \frac{\partial \alpha_1}{\partial \xi_2}\dot\xi_2 - \sum_{i=0,1}\frac{\partial \alpha_1}{\partial v_{i2}}\dot v_{i2} - \sum_{i=0}^{3}\frac{\partial \alpha_1}{\partial f_{i2}}\dot f_{i2} \tag{29b}$$

can be calculated. Note $\dot z_2$ involves $\dot{\hat\rho} = -\dot{\tilde\rho}$ and $\dot{\hat\theta} = -\dot{\tilde\theta}$ which are still unknown. Using our knowledge of circuit architectures we can simultaneously and very easily solve for $\dot{\hat\rho}$ and $\dot{\hat\theta}$ and $\dot z_2$. For this purpose, we are going to identify the augmented error vector **E** with the current through single-port and multi-port inductive elements[7]. Then the terms on the LHS of equation (23a) involving weighted derivatives of the error vector could be identified with the voltages across various ports of those inductive elements. The weights need to be positive or positive definite (in the case of multi-port inductive elements). Then the square matrix on the RHS of equation (25a) should correspond to the negative of the impedance matrix $\hat{\mathbf{Z}}_p$ of a LTI or LTV or NLTI or NLTV static, passive multi-port network. In its simplest form, $\hat{\mathbf{Z}}_p$ could be a sum of a diagonal matrix with terms that are always positive and an anti-metric matrix. Let us assume such a circuit-architecture and first look at the plant error dynamics (equation 9a or the four equations in 25a). They are not functions of $\tilde\rho$ or $\tilde\theta$ making the matrix elements $\hat z_{pij}$ $\left(i = 3 \ to\ 6 \ and \ j = 7 \ to \ 14\right)$ zero. That in turn means $\dot{\tilde\rho}$ and $\dot{\tilde\theta}$ equations should have zeros $\left(\hat z_{pij} = 0\right)$ in the matrix positions $i = 7 \ to \ 14 \ and \ j = 3 \ to \ 6$ corresponding to the plant error vector ε. Further, the equations for $\dot{\hat\rho}$ and $\dot{\hat\theta}$ cannot involve $\tilde\rho$ or $\tilde\theta$ since those parameters are not available to us. Hence we will fill the matrix with zeros corresponding to those variables $\hat z_{pij}$ $\left(i = 7 \ to \ 14 \ and \ j = 7 \ to \ 14\right)$. The equation for $\dot z_1$ also fixes some of the terms in the matrix $\hat{\mathbf{Z}}_p$ as shown in (25b). Thus we end up with the following possibility for the remaining unknown dynamics from a purely circuits' perspective as:

$$c_2\,\dot z_2 = -m_{12}\,z_1 - m_{22}\,z_2 + \sum_{j=3}^{6} m_{2j}\,\varepsilon_{j-2} + m_{27}\,\tilde\rho + \bar m_2\tilde\theta \tag{30a}$$

---

[7] The variables z_1, z_2, $\tilde\rho$ will be associated with one-port elements and the vectors ε and $\tilde\theta$ with multi-port elements.

$$c_7 \dot{\tilde{\rho}} = -m_{17} z_1 - m_{27} z_2 = b_2(\dot{y}_d + \bar{\alpha}_1) z_1 - m_{27} z_2 \tag{30b}$$

$$\bar{\mathbf{L}}\dot{\tilde{\theta}}_{7\times1} = -\bar{\mathbf{L}}\dot{\hat{\theta}}_{7\times1} = -\bar{m}_1^t z_1 - \bar{m}_2^t z_2 \tag{30c}$$

where again the equation for $\dot{\tilde{\theta}}$ has been written using the property of $\hat{\mathbf{Z}}_p$. Note m_{12}, m_{17}, $\bar{m}_1$ are fixed based on $\dot{z}_1$, and we still need to fix $c_i\ (>0)$, $m_{ij}\ (m_{22} \ge 0)$, $\bar{m}_2$ [8]. The equation for $\dot{\tilde{\rho}}$ involves b_2 that is not available. The need for that term could be avoided if we let $c_7 = \gamma\,|b_2|\ (\gamma > 0)$, make the mild assumption that the sign of b_2 is known, and $m_{27} = 0$. Then the dynamics for $\dot{\tilde{\rho}} = -\dot{\hat{\rho}}$ reduces to:

$$\dot{\tilde{\rho}} = -\dot{\hat{\rho}} = \mathrm{sgn}[b_2](\dot{y}_d + \bar{\alpha}_1) z_1/\gamma \tag{31a}$$

which can be implemented. Equating the expression for $\dot{z}_2$ obtained from the plant (27) with that from circuit considerations (30a) along with the equations for $\dot{\hat{\theta}}$ and $\dot{\hat{\rho}}$ (30c and 31) and noting that the input variable $u(t)$ cannot include the unknown variables ε_1 *to* ε_4, $\tilde{\rho}$, $\tilde{\theta}$, we get the following solution:

$$\begin{aligned} &m_{23} = m_{25} = m_{26} = m_{27} = 0;\ \ m_{24} = c_2\frac{\partial\alpha_1}{\partial y};\ \ \bar{m}_2 = c_2\frac{\partial\alpha_1}{\partial y}\omega; \\ &u(t) = \beta - \frac{m_{12}}{c_2}z_1 - \frac{m_{22}}{c_2}z_2 + \hat{\rho}\,\ddot{y}_d - \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}} - \frac{\partial\alpha_1}{\partial\hat{\rho}}\dot{\hat{\rho}} \end{aligned} \tag{32}$$

From (30) to (32) and based on the known property of the matrix $\hat{\mathbf{Z}}_p$, we can fix the values of fn_{ij} in the plant error dynamics as:

$$fn_{11} = fn_{12} = fn_{31} = fn_{32} = fn_{41} = fn_{42} = 0;\ \ fn_{21} = 1;\ \ fn_{22} = m_{24}; \tag{33}$$

leading to the total error dynamics as shown in equation (25).

It is very easy to show that the augmented error dynamics has the origin as the globally UAS equilibrium point using circuit interpretations. As noted before the error vector $\mathbf{E}$ would be associated with the currents in the various inductors (or a single multi-port inductor). The vector $\hat{\mathbf{L}}\dot{\mathbf{E}}$ (with $\hat{\mathbf{L}}$ the mutual inductance and a positive definite matrix) on the LHS of equation (25) corresponds to the voltage across that inductor. The inductor has no stored energy or in a state of relaxation at the origin. The passive network represented by the impedance matrix $\hat{\mathbf{Z}}_p$ can only consume power and there is no other energy

[8] The positive values are necessary since those terms will be associated with the values of the inductors.

source in the circuit representation. Therefore the error vector will go to the origin as $t \to \infty$ from any initial value. In fact, we can write from circuit considerations one Lyapunov function (the true stored energy function) and its derivative (as the negative of the power consumed by the various resistors) as:

$$
\begin{aligned}
V &= 0.5\,\mathbf{E}^t\,\hat{\mathbf{L}}\,\mathbf{E} \ge 0 \\
dV/dt\big|_{sys\ trajectory} &= -\mathbf{E}^t\,\hat{\mathbf{Z}}_p\,\mathbf{E} = -m_{11}\,z_1^2 - m_{22}\,z_2^2 - \sum_{i=1}^{4} z_{pii}\,\mathrm{e}_i^2 \le 0
\end{aligned}
\tag{34}
$$

In fact, by considering various circuit elements (LTI to NLTV) and the dynamics from circuits made from such elements along with proper attention to the constraints imposed by the particular application, we can arrive easily at numerous complex solutions that would be very difficult if not impossible to arrive at using the analytical approach. In this particular application, we need the origin to be a globally UAS equilibrium point under the constraint that many of the state variables are not available. One general form of the dynamics that include a new dynamic control variable $z(t)$, similar to the integrator back-stepping approach, would be of the form:

$$
\begin{bmatrix}
\dot{z}_1 \\
c_2[\mathbf{E}]\,\dot{z}_2 \\
\Sigma_{j=1}^4 L_{1j}\,\dot{\mathrm{e}}_j \\
\Sigma_{j=1}^4 L_{2j}\,\dot{\mathrm{e}}_j \\
\Sigma_{j=1}^4 L_{3j}\,\dot{\mathrm{e}}_j \\
\Sigma_{j=1}^4 L_{4j}\,\dot{\mathrm{e}}_j \\
c_7[\mathbf{E}]\,\dot{\tilde{r}} \\
\bar{\mathbf{L}}[\mathbf{E}]\,\dot{\tilde{\mathrm{q}}}_{7\times1} \\
c_8[\mathbf{E}]\,z(t)
\end{bmatrix}
=
\begin{bmatrix}
-m_{11} & m_{12} & 0 & 1 & 0 & 0 & m_{17} & (\bar{m}_1)_{1\times7} & m_{18}[\mathbf{E}] \\
-m_{12} & -m_{22} & 0 & m_{24} & 0 & 0 & 0 & (\bar{m}_2)_{1\times7} & m_{28}[\mathbf{E}] \\
0 & 0 & -z_{p11} & -z_{p12} & -z_{p13} & -z_{p14} & 0 & 0_{1\times7} & 0 \\
-1 & -m_{24} & -z_{p21} & -z_{p22} & -z_{p23} & -z_{p24} & 0 & 0_{1\times7} & 0 \\
0 & 0 & -z_{p31} & -z_{p32} & -z_{p33} & -z_{p34} & 0 & 0_{1\times7} & 0 \\
0 & 0 & -z_{p41} & -z_{p42} & -z_{p43} & -z_{p44} & 0 & 0_{1\times7} & 0 \\
-m_{17} & 0 & 0 & 0 & 0 & 0 & 0 & 0_{1\times7} & 0 \\
-(\bar{m}_1^t)_{1\times7} & (\bar{m}_2)_{1\times7} & 0_{7\times1} & 0_{7\times1} & 0_{7\times1} & 0_{7\times1} & 0_{7\times1} & 0_{7\times7} & 0 \\
-m_{18}[\mathbf{E}] & -m_{28}[\mathbf{E}] & 0 & 0 & 0 & 0 & 0 & 0 & -m_{88}
\end{bmatrix}
\begin{bmatrix}
i_{R1}[z_1] \\
i_{R2}[z_2] \\
\mathrm{e}_1 \\
\mathrm{e}_2 \\
\mathrm{e}_3 \\
\mathrm{e}_4 \\
\tilde{r} \\
\tilde{\mathrm{q}} \\
i_R[z]
\end{bmatrix}
\tag{35}
$$

In the above equation, using circuits' interpretations we have introduced a number of terms that are functions of the error vector **E**. Terms such as $c_j[\mathbf{E}]$ have to be positive valued and bounded, $\bar{\mathbf{L}}[\mathbf{E}]$ a TV bounded PD matrix, $i_{Rj}\left[z_j\right]$ represents a resistor mapping (confined to the first- and the third-quadrant and passes through the origin) and $m_{ij}[\mathbf{E}]$ any real function. We can further constrain their values based on how the dynamics should behave. For example, we can use $m_{28}[\mathbf{E}]$ to make the input $u(t)$ bounded. From the above equation, it should be clear how powerful the method is.

Simulations:

1) First, the simple model:

Assume various constants, y_d as given in Ref. {1, 2]. Assume the **k** vector given for the error dynamics and find L and Zp from steps given below equation (7).

$\mathbf{A}_e = -\mathbf{L}^{-1}\mathbf{Z}_p$ obtain $\mathbf{L}$ from $\left(\mathbf{L}\mathbf{A}_e + \mathbf{A}_e^t\mathbf{L}\right) = -2\,\mathbf{Z}_R$,

$\mathbf{Z}_p$ is found from $\mathbf{Z}_p = -\mathbf{L}\,\mathbf{A}_e$.

Assume nonzero initial values for all paramter estimates. assume all other dynamic values zero for the first round

Find $z_1(t) = y(t) - y_d(t)$

$$\alpha_1 = \hat\rho\left(\xi_2 + \bar\omega\hat\theta + m_{11} z_1\right);\ \bar\alpha_1 = \alpha_1/\hat\rho \qquad (21)$$

$$m_{23} = m_{25} = m_{26} = m_{27} = 0;\ \ m_{24} = -c_2\frac{\partial\alpha_1}{\partial y};\ \ \bar m_2 = -c_2\frac{\partial\alpha_1}{\partial y}\omega;$$

$$u(t) = \beta - \frac{m_{12}}{c_2} z_1 - \frac{m_{22}}{c_2} z_2 + \dot{\hat\rho}\,\dot y_d + \frac{\partial\alpha_1}{\partial\hat\theta}\dot{\hat\theta}$$

$$z_2 = v_{22} - \alpha_1 - \hat\rho\,\dot y_d$$

Then put the plant dynamics (2 state space equations;

$$M\ddot y(t) + C\dot y(t) + k\,y(t) = u(t) + F_d(t)$$
$$F_d(t) = Q\sin(\omega t) \qquad (1)$$

the calculatable part of the dynamics (another 4 plus 7x8 for a total of 60 state equations)

$$\dot\xi = \mathbf{A}_e\,\xi + \mathbf{k}\,y - \mathbf{L}^{-1}\mathbf{f}[z_1, z_2] \qquad (11a)$$

$$\dot\Omega = \mathbf{A}_e\,\Omega + \mathbf{F}[y, u] \qquad (11b)$$

Eight parameter update equations derived from equation (25a)

$$\dot{\hat\rho} = -\mathrm{sgn}[b_2](\dot y_d + \bar\alpha_1)\,z_1/\gamma \qquad (31)$$

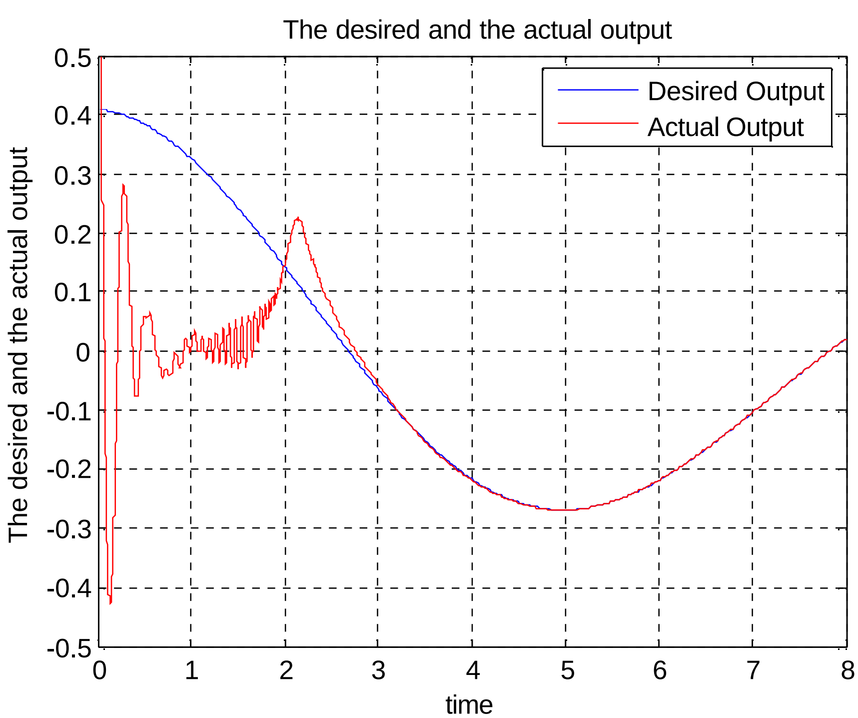
$$\bar{\mathbf{L}}\,\dot\theta_{7\times1} = -\bar m_1^t z_1 - \bar m_2^t z_2 \qquad (30c)$$

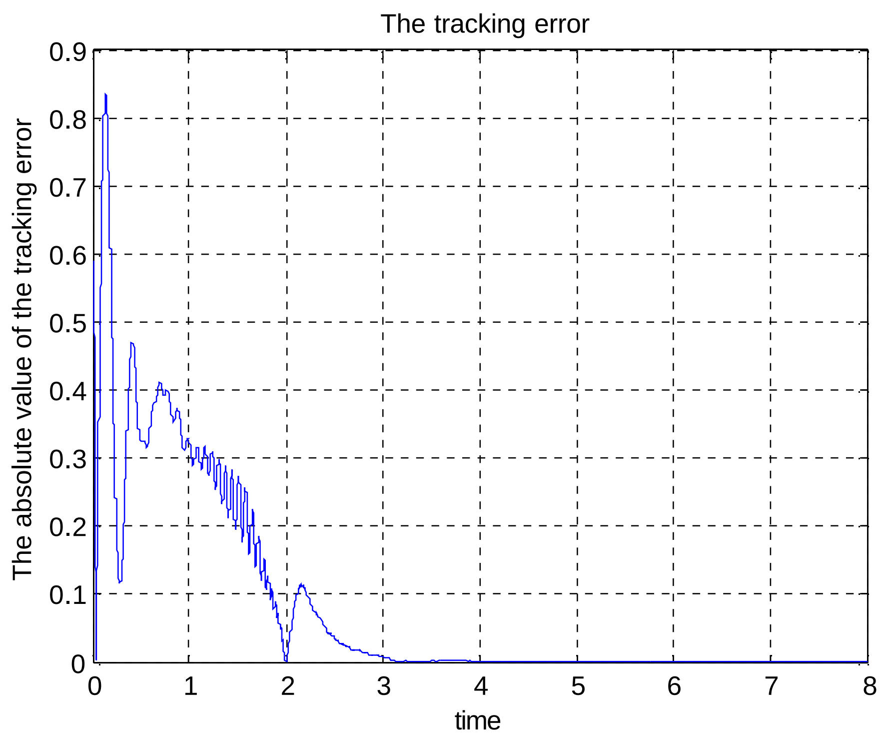
for a total of 70 state equations)

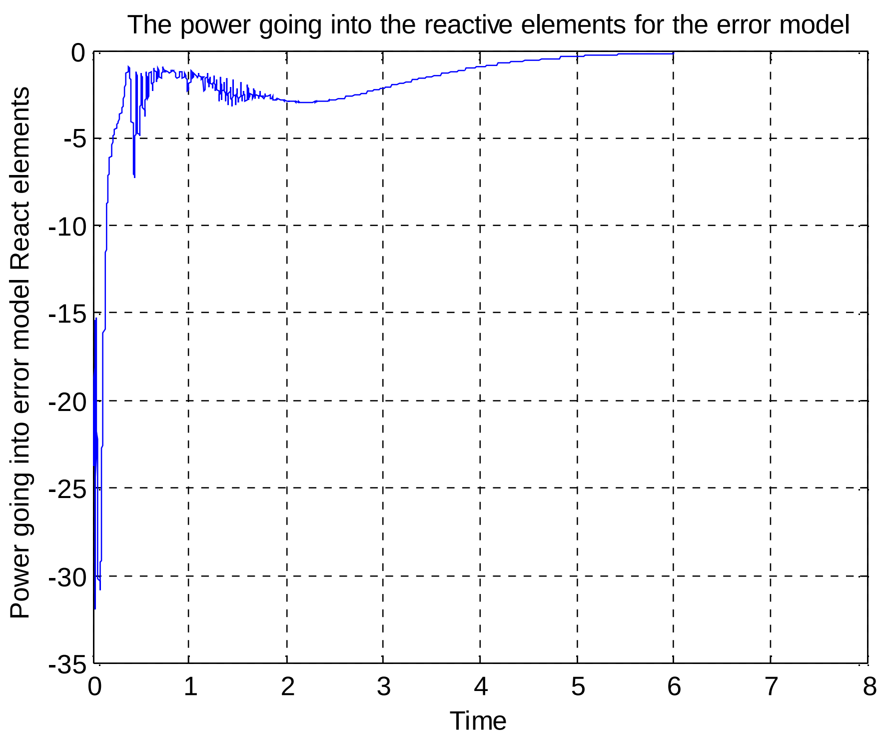
You might also include the total error dynamics (14 equations in equation 25a) into this dynamics (total of 84 state equations) and put them in a single ODE45 function call. You would be including the augmented error dynamics 25a just to make sure that it works the way it is supposed to do – error goes to zero as time increases but you should not be using any state values from the error in any other equations.

If I write the program, I will define all variable and vectors as defined in the paper (using same names etc.) and calculate the values as given by the various equations (even if it becomes computationally inefficient). For example, all the state variables would come as a single vector from ODE45; pull them out separately to form **x** (the plant state values; 2 elements), x (4 variables), Ω (4 x 7 matrix) etc. This way, if there is any mistake, it would be easy to fix the error or modify the equations (if there is any error in their formulation).

Simulation Results:




The tracking error
The absolute value of the tracking error
time
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0 1 2 3 4 5 6 7 8


The power going into the reactive elements for the error model
Power going into error model React elements
Time
0
-5
-10
-15
-20
-25
-30
-35
0 1 2 3 4 5 6 7 8

2. **Modified adaptive equations:**

Assume some nonlinear and TV mappings as given in (35) (with out z(t)), derive the new set of equations and repeat step # 1.

3. **Bounded input Model:**

We will introduce the new state variable z(t). Derive the new set of equations, use $m_{28}[\mathbf{E}]$ to control the input magnitude and use the same value in adapting z(t). And simulate the system again.